Assessing Model Risk on Dependence in High Dimensions

Carole Bernard (University of Waterloo)
& Steven Vanduffel (Vrije Universiteit Brussel)

Alexander von Humboldt Stiftung/Foundation

University of Waterloo

WATRISQ
Waterloo Research Institute in Insurance, Securities & Quantitative Finance

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Risk Aggregation and Diversification

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$$std(X_1 + X_2) = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$$

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- This is not the case for instance for Value-at-Risk.

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- Basel II, Solvency II, Swiss Solvency Test, US Risk Based Capital, Canadian Minimum Continuing Capital and Surplus Requirements (MCCSR): all recognize partially the benefits of diversification and aggregating risks may decrease the overall capital.
- But they also recognize the difficulty to find an adequate model to aggregate risks.
  - **Var-covar** approach based on a correlation matrix: correlation is a poor measure of dependence, issue with micro-correlation, correlation 0 does not mean independence, problem of tail dependence, correlation is a measure of linear dependence.
  - **Copula approach**, vine models... : very flexible but prone to model risk
  - **Scenario based approach**, including identifying common risk factors and incorporate what you know in the model.

Objectives and Findings

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  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of the portfolio?

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Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of d individual dependent risks.
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- Analytical expressions for these maximum and minimum
- A non-parametric method based on the data at hand.
- Implications:
  - Current regulation is subject to very high model risk, even if one knows the multivariate distribution almost completely.
  - Able to quantify model risk for a chosen risk measure. We can identify for which risk measures it is meaningful to develop accurate multivariate models.

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1. Goal: Assess the risk of a portfolio sum $S = \sum_{i=1}^{d} X_i$.
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3. How about model risk? How wrong can we be?

Choice of the risk measure

- **Variance** of X
- **Value-at-Risk** of X at level $p \in (0,1)$

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geqslant p\} \qquad (1)$$

- **Tail Value-at-Risk** or **Expected Shortfall** of X

$$\mathrm{TVaR}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_u(X)\mathrm{d}u \qquad p \in (0,1)$$

and $p \to \mathrm{TVaR}_p$ is continuous.

- **Left Tail Value-at-Risk** of X

$$\mathrm{LTVaR}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_u(X)\mathrm{d}u$$

Assessing Model Risk on Dependence with $d = 2$ Risks

definition: Convex order

X is smaller in convex order, $X \prec_{cx} Y$, if for all convex functions f

$$E[f(X)] \leqslant E[f(Y)]$$

Assume first that we trust the marginals $X_i \sim F_i$ but that we have no trust about the dependence structure between the X_i (copula).

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In two dimensions, assessing model risk on $\rho(S)$ is linked to the Fréchet-Hoeffding bounds or "extreme dependence".

$$F_1^{-1}(U) + F_2^{-1}(1-U) \prec_{cx} X_1 + X_2 \prec_{cx} F_1^{-1}(U) + F_2^{-1}(U)$$

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- For risk measures preserving convex order ($\rho(S) = var(S)$, $\rho(S) = TVaR(S)$), for $U \sim \mathcal{U}(0,1)$

$$\rho\left(F_1^{-1}(U) + F_2^{-1}(1-U)\right) \leqslant \rho(S) \leqslant \rho\left(F_1^{-1}(U) + F_2^{-1}(U)\right)$$

This does not apply to Value-at-Risk.

Assessing Model Risk on Dependence with $d \geqslant 3$ Risks

- The Fréchet upper bound corresponds to the comonotonic scenario:

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- In d dimensions
  - Puccetti and Rüschendorf (2012, JCAM): algorithm (RA) to approximate bounds on functionals.
  - Embrechts, Puccetti, Rüschendorf (2013, JBF): application of the RA to find bounds on VaR
  - Bernard, Jiang, Wang (2014, IME): explicit form of the lower bound for convex risk measures of an homogeneous sum.
- Issues
  - bounds are generally very wide
  - ignore all information on dependence.

Incorporating Partial Information on Dependence

- With $d = 2$:
  - subset of bivariate distribution with given measure of association Nelsen et al. (2001 Commun. Stat Theory Methods, 2004, JMVA)
  - bounds for bivariate distribution functions when there are constraints on the values of its quartiles (Nelsen et al. (2004)).
  - 2-dim copula known on a subset of $[0, 1]^2 \Rightarrow$ find "improved Fréchet bounds", Tankov (2011, JAP), Bernard et al. (2012, JAP) and Sadooghi-Alvandi et al. (2013, Commun. Stat. Theory Methods).
- With $d \geqslant 3$: Bounds on the VaR of the sum
  - with known bivariate distributions: Embrechts, Puccetti and Rüschendorf (2013)
  - with the variance of the sum (WP with Rüschendorf,Vanduffel)
  - with higher moments (WP with Denuit, Vanduffel)

Our assumptions

Let $(X_1, X_2, ..., X_d)$ be some random vector of interest. Let $\mathcal{F} \subset \mathbb{R}^d$ ("trusted" or "fixed" area) and $\mathcal{U} = \mathbb{R}^d \backslash \mathcal{F}$ ("untrusted" area). We assume that we know

(i) the marginal distribution F_i of X_i on $\mathbb{R}$ for $i = 1, 2, ..., d$,

(ii) the distribution of $(X_1, X_2, ..., X_d) \mid \{(X_1, X_2, ..., X_d) \in \mathcal{F}\}$.

(iii) $P\left((X_1, X_2, ..., X_d) \in \mathcal{F}\right)$

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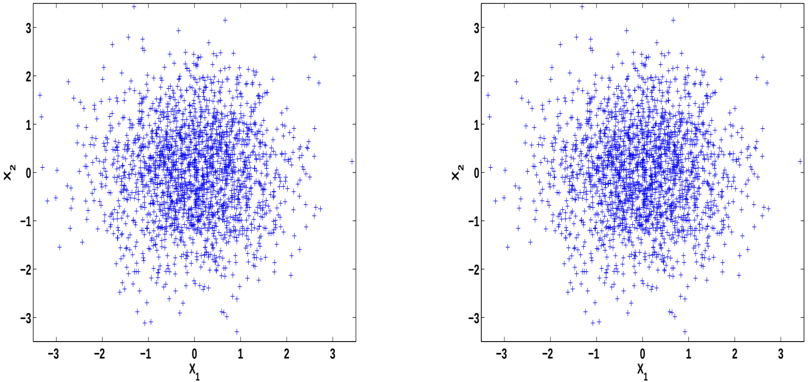
(iii) $P((X_1, X_2, ..., X_d) \in \mathcal{F})$

- The joint distribution of $(X_1, X_2, ..., X_d)$ is thus completely specified if $\mathcal{F} = \mathbb{R}^d$ and $\mathcal{U} = \emptyset$.
- When only marginals are known: $\mathcal{U} = \mathbb{R}^d$ and $\mathcal{F} = \emptyset$.
- **Our Goal:** Find bounds on $\rho(S) := \rho(X_1 + ... + X_d)$ when $(X_1, ..., X_d)$ satisfy (i), (ii) and (iii).

Illustration with marginals N(0,1)



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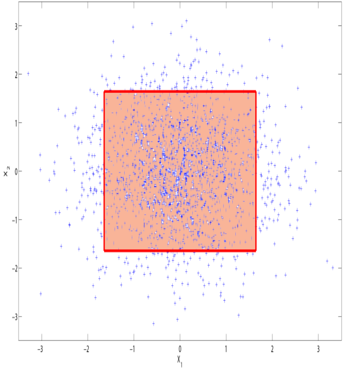

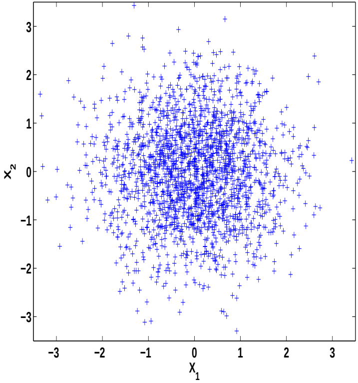

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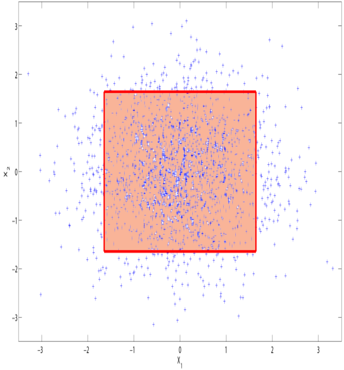

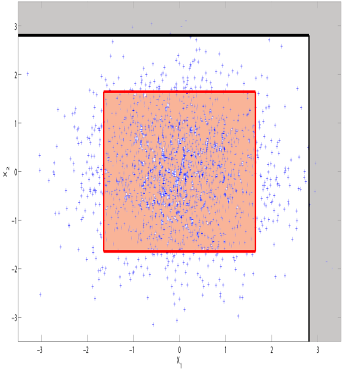

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$$\mathcal{F} = \bigcup_{k=1}^{2} \{X_k > q_p\} \bigcup \mathcal{F}_1$$

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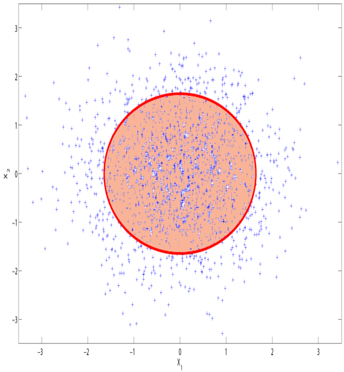
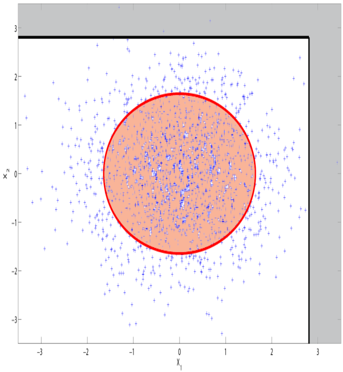


$\mathcal{F}_1$ =contour of MVN at β



$$\mathcal{F} = \bigcup_{k=1}^{2} \{X_k > q_p\} \bigcup \mathcal{F}_1$$

Model Risk

Assume $(X_1, X_2, ..., X_d)$ satisfies (i), (ii) and (iii) and use a risk measure $\rho(\cdot)$. Define

$$\rho_{\mathcal{F}}^{+} := \sup \left\{ \rho\left(\sum_{i=1}^{d} Y_i\right)\right\}, \quad \rho_{\mathcal{F}}^{-} := \inf \left\{ \rho\left(\sum_{i=1}^{d} Y_i\right)\right\}$$

where the supremum and the infimum are taken over all other (joint distributions of) random vectors $(Y_1, Y_2, ..., Y_d)$ that agree with (i), (ii) and (iii).

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where the supremum and the infimum are taken over all other (joint distributions of) random vectors $(Y_1, Y_2, ..., Y_d)$ that agree with (i), (ii) and (iii).

"Model risk of underestimation" of $\rho(\sum X_i)$ in some chosen benchmark model:

$$\frac{\rho^+_{\mathcal{F}} - \rho(\sum_{i=1}^{n} X_i)}{\rho^+_{\mathcal{F}}}$$

"Model risk of overestimation" of $\rho(\sum X_i)$:

$$\frac{\rho(\sum_{i=1}^{n} X_i) - \rho^-_{\mathcal{F}}}{\rho^-_{\mathcal{F}}}$$

Technical Contributions

1. The first approach is practical: an algorithm to approximate the sharp bounds $\rho_{\mathcal{F}}^{-}$ and $\rho_{\mathcal{F}}^{+}$ performed directly using the data at hand (without fitting a model): model risk can be assessed in a fully **non-parametric** way: Use of the rearrangement algorithm of Puccetti and Rüschendorf (2012) and Embrechts et al. (2013).
2. The second approach provides theoretical bounds, which can be directly computed within a model (**Monte Carlo**) but may not be sharp.

First Approach

Approximation of Bounds

(for variance and TVaR)

Non-parametric Approach

- N observations of the d-dimensional vector $(x_{i1}, x_{i2}, ..., x_{id})$ for $i = 1, ..., N$. The corresponding $N \times d$ matrix:

$$M = (x_{ij})_{i,j}$$

- Each observation $(x_{i1}, x_{i2}, ..., x_{id})$ occurs with probability $\frac{1}{N}$ naturally (possibly involving repetitions).
- M contains enough data for an accurate description of the marginal distributions of X_k $(k = 1, 2, ..., d)$
- Define S_N by $S_N(i) = \sum_{k=1}^{d} x_{ik}$ for $(i = 1, 2, ..., N)$. S_N can be seen as a random variable that takes the value $S_N(i)$ in "state" i for $i = 1, 2, ..., N$.

Goal: Find (sharp) bounds on the risk measure applied to S_N.

Example of M:

$N = 8$ observations, $d = 3$ dimensions
and 3 observations trusted ($\ell_f = 3$, $p_f = 3/8$)

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad S_N = \begin{bmatrix} 8 \\ 3 \\ 5 \\ 3 \\ 8 \\ 4 \\ 4 \\ 9 \end{bmatrix}$$

- The matrix M is split into two parts: $\mathcal{F}_N$: trusted observations, $\mathcal{U}_N$: "untrusted" part.
- Rearranging the values x_{ik} $(i = 1, 2, ..., N)$ within the $k-$th column does not affect the marginal distribution X_k but only changes the observed dependence.
- ℓ_f : number of elements in $\mathcal{F}_N$, ℓ_u : number of elements in $\mathcal{U}_N$

$$N = \ell_f + \ell_u.$$

- M has ℓ_f grey rows and ℓ_u white rows.
- S_N^f and S_N^u consist of sums in $\mathcal{F}_N$ and $\mathcal{U}_N$.

Example: $N = 8$, $d = 3$ with 3 observations trusted ($\ell_f = 3$)

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad S_N = \begin{bmatrix} 8 \\ 3 \\ 5 \\ 3 \\ 8 \\ 4 \\ 4 \\ 9 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 10 \\ 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

Bounds on Variance (or TVaR) - Maximum variance

Maximum in convex order:

upper Fréchet bound, comonotonic scenario

- To maximize the variance of S_N: comonotonic scenario on $\mathcal{U}_N$, and the corresponding values of the sums are exactly the values $\tilde{s}_i$ $(i = 1, 2, ..., \ell_u)$ in S_N^u.
- The upper bound on variance is then computed as

$$\frac{1}{N}\left(\sum_{i=1}^{\ell_f}(s_i - \bar{s})^2 + \sum_{i=1}^{\ell_u}(\tilde{s}_i - \bar{s})^2\right) \qquad (2)$$

where the average sum $\bar{s} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{d} x_{ij}$

Example: Maximum Variance

With the matrix M of observations

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 10 \\ 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

The average sum is $\bar{s} = 5.5$. The maximum variance is equal to

$$\frac{1}{8}\left(\sum_{i=1}^{3}(s_i - \bar{s})^2 + \sum_{i=1}^{5}(\tilde{s}_i^c - \bar{s})^2\right) \approx 8.75$$

Bounds on Variance (or TVaR) - Minimum variance

Minimum in convex order

The rearrangement algorithm (RA) of Puccetti & Rüschendorf, 2012 aims to obtain sums that are "smallest possible" (for convex order).

Idea of the RA

- Columns of M are rearranged such that they become anti-monotonic with the sum of all other columns "until convergence is reached".

$$\forall k \in \{1, 2, ..., n\}, X_k \textit{ antimonotonic with } \sum_{j \neq k} X_j$$

- Note that after each step, $var\left(X_k^a + \sum_{j\neq k} X_j\right) \leqslant var\left(X_k + \sum_{j\neq k} X_j\right)$ where X_k^a is antimonotonic with $\sum_{j\neq k} X_j$

Example: Minimum Variance

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 10 \\ 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

Example: Minimum Variance

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For the minimum variance, construct convex smallest distribution for S_N^u (ideally constant, "joint mixability") $\Rightarrow$ RA on U_N

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The minimum variance is

$$\frac{1}{8}\left(\sum_{i=1}^{3}(s_i - \bar{s})^2 + \sum_{i=1}^{5}(\tilde{s}_i^m - \bar{s})^2\right) \approx 2.5$$

Second Approach

Model Risk Analytical Bounds

(for variance and TVaR)

Some Notation

- Define $p_f := P(\mathbb{I} = 1)$ and $p_u := P(\mathbb{I} = 0)$ where

$$\mathbb{I} := \mathbb{1}_{(X_1, X_2, ..., X_d) \in \mathcal{F}} \tag{3}$$

- Let $U \sim \mathcal{U}(0,1)$ independent of the event "$(X_1, X_2, ..., X_d) \in \mathcal{F}$" (so U is independent of $\mathbb{I}$).
- Define $(Z_1, Z_2, ..., Z_d)$ by

$$Z_i = F^{-1}_{X_i|(X_1, X_2, ..., X_d) \in \mathcal{U}}(U), \qquad i = 1, 2, ..., d \tag{4}$$

- All Z_i ($i = 1, 2, ..., d$) are increasing in U and thus $(Z_1, Z_2, ..., Z_d)$ is comonotonic with known distribution.

Bounds on Variance

Theorem (Bounds on the variance of $\sum_{i=1}^{d} X_i$)

Let $(X_1, X_2, ..., X_d)$ that satisfies properties (i), (ii) and (iii) and let $(Z_1, Z_2, ..., Z_d)$ and $\mathbb{I}$ as defined before.

$$var\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} E(Z_i)\right) \leqslant var\left(\sum_{i=1}^{d} X_i\right)$$
$$\leqslant var\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} Z_i\right)$$

Example

- Assume $d = 20$.
- (i) $(X_1, ..., X_{20})$ is a random vector with N(0,1) marginals.
- (ii) $(X_1, ..., X_{20})$ follows a multivariate standard normal distribution with correlation parameter (pairwise correlation) ρ on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d$$

  (for some $\beta < 50\%$) where q_γ is the quantile of N(0,1) at level γ.
- $\beta = 0\%$: no uncertainty
- $\beta = 50\%$ full uncertainty

Numerical Results

$\mathcal{F}=[q_\beta, q_{1-\beta}]^d$	$\mathcal{U}=\emptyset$ $\beta=0\%$	$\beta=0.05\%$	$\beta=0.5\%$	$\mathcal{U}=\mathbb{R}^d$ $\beta=50\%$
$\rho=0$	4.47	(4.4 , 5.65)	(3.89 , 10.6)	(0 , 20)
$\rho=0.1$	7.62	(7.41 , 8.26)	(6.23 , 11.7)	(0 , 20)

- First column: standard deviation of $\sum_{i=1}^{20} X_i$ under the assumption of multivariate normality (no dependence uncertainty, i.e., $\mathcal{U}=\emptyset$).
- Lower and upper bounds of the standard deviation of $\sum_{i=1}^{20} X_i$ are reported as pairs $(\rho_{\mathcal{F}}^-, \rho_{\mathcal{F}}^+)$ for various confidence levels β.
- 3,000,000 simulations: all digits reported in the table are significant.

Observations

- Impact of model risk on the standard deviation can be substantial even when the joint distribution $(X_1, ..., X_d)$ is almost perfectly known (β close to 0, p_u close to 0).

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- $\beta = 0.05\%$ and $\rho = 0$. In this case, $p_u = 1 - 0.999^{20} \approx 0.02$. Here, using a multivariate normal assumption might underestimate the standard deviation by (5.65-4.47)/4.47=26.4% and overestimate it by (4.47-4.4)/4.4=1.6%.
- Thus the multivariate normality does not seem to be a prudent assumption: more likely to underestimate risk than to overestimate it.

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- $\beta = 0.05\%$ and $\rho = 0$. In this case, $p_u = 1 - 0.999^{20} \approx 0.02$. Here, using a multivariate normal assumption might underestimate the standard deviation by (5.65-4.47)/4.47=26.4% and overestimate it by (4.47-4.4)/4.4=1.6%.
- Thus the multivariate normality does not seem to be a prudent assumption: more likely to underestimate risk than to overestimate it.
- Adding partial information on dependence (ie when $\beta < 50\%$) reduces the unconstrained bounds ($\beta = 50\%$).
- when $\beta = 0.5\%$ and $\rho = 0$, $p_u = 1 - 0.99^{20} \approx 0.18$ and the unconstrained upper bound for the standard deviation shrinks by approximately 50% (it decreases from 20 to 10.6).

Bounds on TVaR or any risk measure satisfying convex order

Theorem (Bounds on the TVaR of $\sum_{i=1}^{d} X_i$)

Assume $(X_1, X_2, ..., X_d)$ satisfies (i), (ii) and (iii), and let $(Z_1, Z_2, ..., Z_d)$ and $\mathbb{I}$ as defined before.

$$TVaR_p\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} E(Z_i)\right) \leqslant TVaR_p\left(\sum_{i=1}^{d} X_i\right)$$

$$\leqslant TVaR_p\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} Z_i\right)$$

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$$\leqslant TVaR_p\left(\mathbb{I}\sum_{i=1}^{d} X_i + (1-\mathbb{I})\sum_{i=1}^{d} Z_i\right)$$

- Same example with the standard multivariate model as benchmark.
- Conclusions are similar to the variance.

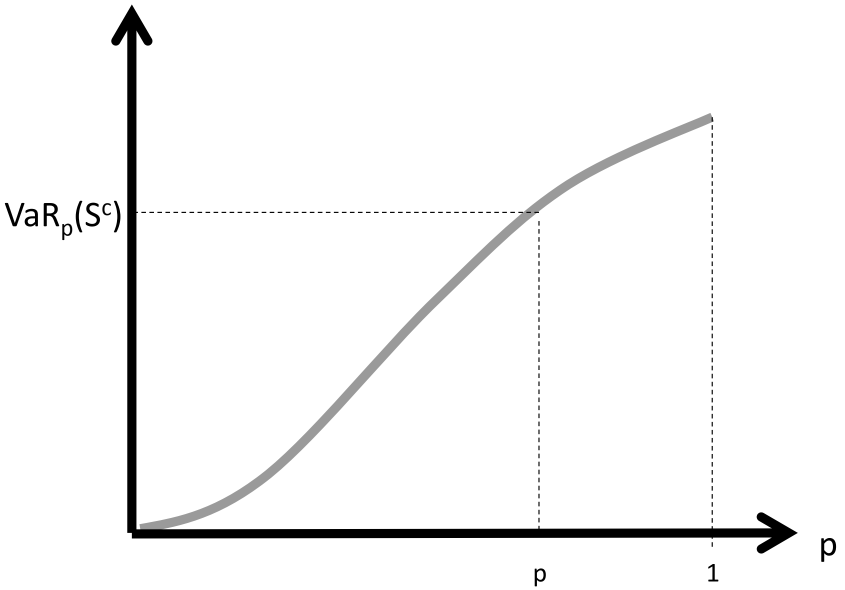
First & Second Approach

Bounds on Value-at-Risk

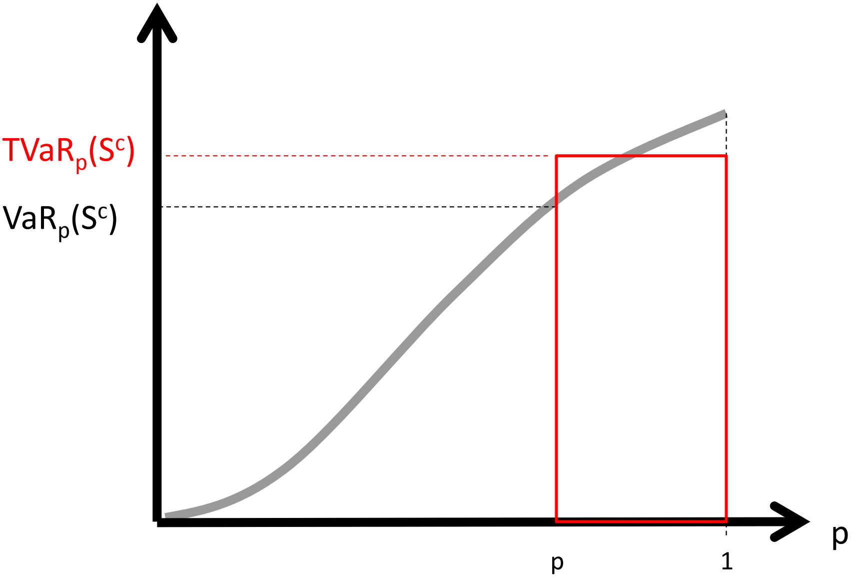
Bounds on Value-at-Risk

Previous approach works for all risk measures that satisfy convex order... But not for Value-at-Risk(S)

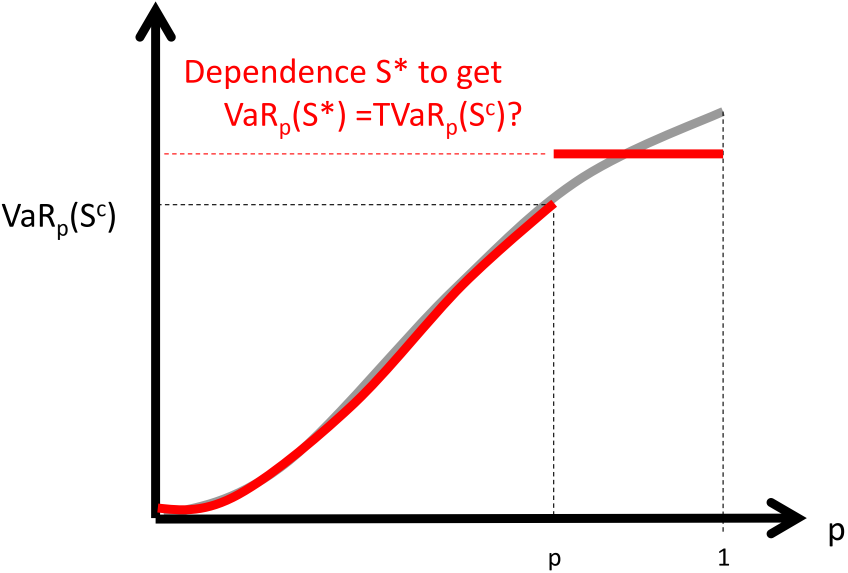
- to maximize VaR_p, the idea is to change the comonotonic dependence of Z_i such that the sum is constant beyond the (comonotonic) VaR level
- to minimize VaR_p, the idea is to change the comonotonic dependence of Z_i such that the sum is constant in the left tail, below the (comonotonic) VaR level (or lowest variance)

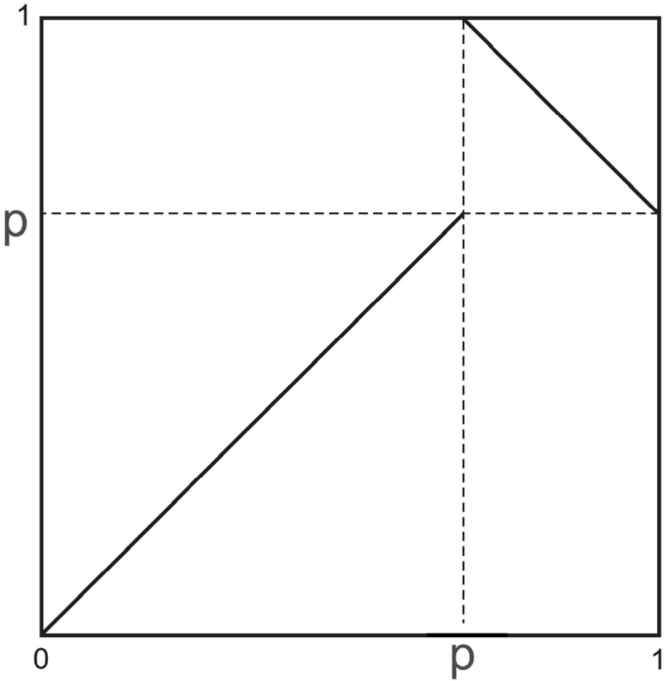


$VaR_p(S^c)$
p
p
1



$TVaR_p(S^c)$
$VaR_p(S^c)$
p
1
p

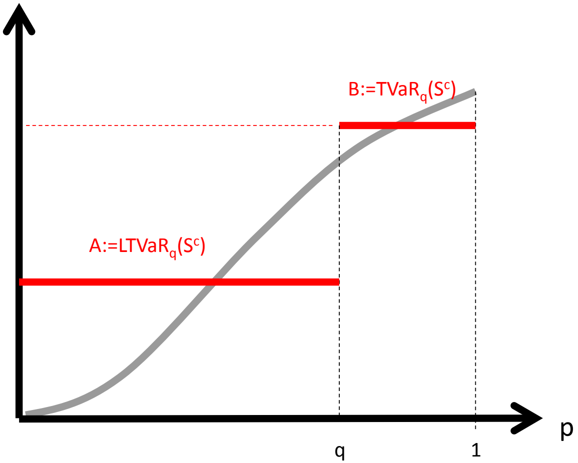

Dependence S* to get
VaR$_p$(S*) =TVaR$_p$(S^c)?
VaR$_p$(S^c)
p
1
p



1
p
0
p
1

Unconstrained Bounds with $X_j \sim F_j$

$$A = LTVaR_q(S^c) \leqslant \mathrm{VaR}_q\left[X_1 + X_2 + ... + X_n\right] \leqslant B = TVaR_q(S^c)$$



Bounds on VaR

Theorem (Constrained VaR Bounds for $\sum_{i=1}^d X_i$)

Assume $(X_1, X_2, ..., X_d)$ satisfies properties (i), (ii) and (iii), and let $(Z_1, Z_2, ..., Z_d)$, U and $\mathbb{I}$ as defined before. Define the variables L_i and H_i as

$$L_i = LTVaR_U(Z_i) \text{ and } H_i = TVaR_U(Z_i)$$

and let

$$m_p := VaR_p\left(\mathbb{I}\sum_{i=1}^d X_i + (1-\mathbb{I})\sum_{i=1}^d L_i\right)$$
$$M_p := VaR_p\left(\mathbb{I}\sum_{i=1}^d X_i + (1-\mathbb{I})\sum_{i=1}^d H_i\right)$$

Bounds on the Value-at-Risk are $m_p \leqslant VaR_p\left(\sum_{i=1}^d X_i\right) \leqslant M_p$.

Value-at-Risk of a Mixture

Lemma

Consider a sum $S = \mathbb{I}X + (1-\mathbb{I})Y$, where $\mathbb{I}$ is a Bernoulli distributed random variable with parameter p_f and where the components X and Y are independent of $\mathbb{I}$. Define $\alpha_* \in [0,1]$ by

$$\alpha_* := \inf\left\{\alpha \in (0,1) \mid \exists \beta \in (0,1) \left\{ \begin{array}{l} p_f\alpha + (1-p_f)\beta = p \\ VaR_\alpha(X) \geqslant VaR_\beta(Y) \end{array} \right.\right\}$$

and let $\beta_* = \frac{p - p_f\alpha_*}{1-p_f} \in [0,1]$. Then, for $p \in (0,1)$,

$$VaR_p(S) = \max\left\{VaR_{\alpha_*}(X), VaR_{\beta_*}(Y)\right\}$$

Applying this lemma, one can prove a more convenient expression to compute the VaR bounds.

Let us define $T := F^{-1}_{\sum_i X_i | (X_1, X_2, \ldots, X_d) \in \mathcal{F}}(U)$.

Theorem (Alternative formulation of the upper bound for VaR)

Assume $(X_1, X_2, \ldots, X_d)$ *satisfies properties (i), (ii) and (iii), and let* $(Z_1, Z_2, \ldots, Z_d)$ *and* $\mathbb{I}$ *as defined before.*

With $\alpha_1 = \max\left\{0, \frac{p+p_f-1}{p_f}\right\}$ *and* $\alpha_2 = \min\left\{1, \frac{p}{p_f}\right\}$,

$\alpha_* := \inf\left\{\alpha \in (\alpha_1, \alpha_2) \mid VaR_\alpha(T) \geqslant TVaR_{\frac{p-p_f\alpha}{1-p_f}}\left(\sum_{i=1}^d Z_i\right)\right\}$

When $\frac{p+p_f-1}{p_f} < \alpha_* < \frac{p}{p_f}$,

$$M_p = TVaR_{\frac{p-p_f\alpha_*}{1-p_f}}\left(\sum_{i=1}^d Z_i\right)$$

The lower bound m_p is obtained by replacing "TVaR" by "LTVaR".

Numerical Results, $\mathcal{F} = [q_\beta, q_{1-\beta}]^d$, $\rho = 0.1$

	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$\beta = 0.05\%$	$\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
p=95%	12.5	(12.2 , 13.3)	(10.7 , 27.7)	(-2.17 , 41.3)
p=99.95%	25.1	(24.2 , 71.1)	(21.5 , 71.1)	(-0.035 , 71.1)

- $\mathcal{U} = \emptyset$: No uncertainty (multivariate standard normal model).
- $3,000,000$ simulations: all digits reported are significant.

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- $\mathcal{U} = \emptyset$: No uncertainty (multivariate standard normal model).
- $3,000,000$ simulations: all digits reported are significant.

- **The risk for an underestimation of VaR is increasing in the probability level used to assess the VaR.**
- When very high probability levels are used in the VaR calculations ($p = 99.95\%$), the constrained bounds are very close to the unconstrained bounds even when there is almost no uncertainty on the dependence ($\beta = 0.05\%$).
- **So despite all the added information on dependence, the bounds are still wide!**

With Pareto risks

Consider $d = 20$ risks distributed as Pareto with parameter $\theta = 3$.

- Assume we trust the independence conditional on being in $\mathcal{F}_1$

$$\mathcal{F}_1 = \bigcap_{k=1}^{d} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$$

where $q_\beta = (1-\beta)^{-1/\theta} - 1$.

$\mathcal{F}_1$	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$\beta = 0.05\%$	$\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
α=95%	16.6	(16 , 18.4)	(13.8 , 37.4)	(7.29 , 61.4)
α=99.95%	43.5	(26.5 , 359)	(20.5 , 359)	(9.83 , 359)

Incorporating Expert's Judgements

Consider $d = 20$ risks distributed as Pareto $\theta = 3$.

- Assume comonotonicity conditional on being in $\mathcal{F}_2$

$$\mathcal{F}_2 = \bigcup_{k=1}^{d} \{X_k > q_p\}$$

Comonotonic estimates of Value-at-Risk
$VaR_{95\%}(S^c) = 34.29$, $VaR_{99.95\%}(S^c) = 231.98$

$\mathcal{F}_2$	$\mathcal{U} = \emptyset$ (Model)	$p = 99.5\%$	$p = 99.9\%$	$p = 99.95\%$
α=95%	16.6	(8.35 , 50)	(7.47 , 56.7)	(7.38 , 58.3)
α=99.95%	43.5	(232 , 232)	(232 , 232)	(180 , 232)

Comparison

Independence within a rectangle $\mathcal{F}_1 = \bigcap_{k=1}^{d} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$

$\mathcal{F}_1$	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$\beta = 0.05\%$	$\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
α=95%	16.6	(16 , 18.4)	(13.8 , 37.4)	(7.29 , 61.4)
α=99.95%	43.5	(26.5 , 359)	(20.5 , 359)	(9.83 , 359)

Comonotonicity when one of the risks is large $\mathcal{F}_2 = \bigcup_{k=1}^{d} \{X_k > q_p\}$

$\mathcal{F}_2$	$\mathcal{U} = \emptyset$ (Model)	$p = 99.5\%$	$p = 99.9\%$	$p = 99.95\%$
α=95%	16.6	(8.35 , 50)	(7.47 , 56.7)	(7.38 , 58.3)
α=99.95%	43.5	(232 , 232)	(232 , 232)	(180 , 232)

Algorithm to approximate sharp bounds

- A detailed algorithm to approximate sharp bounds is given in the paper.
- An application to a portfolio of stocks using market data is also fully developed.

Algorithm to approximate sharp bounds

- From the lemma, the VaR of a mixture is obtained as the maximum of two VaRs.
- At the upper bound, this VaR becomes a TVaR (proposition).
- Compute α_* and find a dependence in the vector $(Z_1, Z_2, ..., Z_d)$ such that

$$\mathrm{VaR}_{\beta_*}\left(\sum_{i=1}^{d} Z_i\right) = \mathrm{TVaR}_{\beta_*}\left(\sum_{i=1}^{d} Z_i\right) \tag{5}$$

  where $\beta_* = \frac{p - p_f \alpha_*}{1 - p_f}$
- This is the spirit of the algorithm... where we find the number of rows to take in the untrusted matrix to apply the RA.

Conclusions

- Assess model risk with partial information and given marginals (by Monte Carlo from the fitted distribution or non-parametrically)
- We provide several ways to choose the trusted area $\mathcal{F}$: d-cube or contours of a multivariate density fitted to data. Open question: how to optimally do so?
- N too small but one believes in fitted marginals then improve the efficiency of the algorithm by re-discretizing using the fitted marginal $\hat{f}_i$.
- Possible to amplify the tails of the marginals if one does not trust the marginals, e.g., apply a distortion to amplify the tails when re-discretizing.
- Additional information on dependence can be incorporated
  - variance of the sum (WP with Rüschendorf,Vanduffel)
  - higher moments (WP with Denuit, Vanduffel)

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